Bayesian estimation of a source term of radiation release with approximately known nuclide ratios

Ondřej Tichý, Václav Šmídl and Radek Hofman

Institute of Information Theory and Automation of the CAS, Prague, Czech Republic

[otichy,smidl,hofman]@utia.cas.cz

1. Introduction

We are concerned with the problem of determination of a source term of an atmospheric release from gamma dose rate (GDR) measurements. Usually, the source term of an accidental release of radiation comprises of a mixture of nuclide. The GDR measurements do not provide a direct information on the source term composition [4]. However, physical properties of respective nuclide (deposition properties, decay half-life) can be used when uncertain information on nuclide ratios is available, e.g. from known reactor inventory.

In its simplest form, the problem can be viewed as a liner system $\mathbf{y} = M\mathbf{x}$ with vector of GDR measurements $\mathbf{y}$ and source-receptor-sensitivity (SRS) matrix M, where solution $\mathbf{x}$ can be found using the least-square method. In reality, this is usually not possible since the system is ill-conditioned and an optimization of a regularized cost function must be used:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{X}} \left\{ ||\mathbf{y} - M\mathbf{x}||_2^2 + \alpha g(\mathbf{x}) \right\}, \quad \text{subject to } \mathbf{x} \geq 0, \tag{1}$$

where the first term minimizes the error of the measurements, and the second term is the regularization term with weight α. Various types of regularization arise for different choices of function $g(\mathbf{x})$.

In this contribution, we aim to use approximate information about ratios of nuclides from which the source term is composed. Specifically, we assume the knowledge of the intervals in which lay ratios of nuclides, [2]. We assume that the interval with ratio for kth nuclide $\mathbf{x}_k$ is

$$a_k \leq \frac{\mathbf{x}_1}{\mathbf{x}_k} \leq b_k, \tag{2}$$

where $\mathbf{x}_1$ is the reference nuclide.

2. Bayesian approach

In Bayesian approach, the task of inferring data from observations is formalized as an update of prior belief about the parameter values updated by the available observations. Formally, our knowledge about the parameter vector $\mathbf{x}$ is described by the probability density function

$$p(\mathbf{x}|\mathbf{y}, M) = \frac{p(\mathbf{y}|\mathbf{x}, M)p(\mathbf{x})}{\int p(\mathbf{y}|\mathbf{x}, M)p(\mathbf{x})d\mathbf{x}} \propto p(\mathbf{y}|\mathbf{x}, M)p(\mathbf{x}), \tag{3}$$

where $p(\mathbf{x})$ is the prior distribution, $p(\mathbf{y}|x, M)$ is the likelihood of the measurements. For the choice of Gaussian models

$$p(\mathbf{y}|\mathbf{x}, M) = \mathcal{N}\left(M\mathbf{x}, \omega^{-1}I_p\right), \quad p(\mathbf{x}|\Omega) = \mathcal{N}\left(0, \Omega^{-1}\right), \tag{4}$$

the result of the Bayes rule (3) is again a Gaussian distribution $p(\mathbf{x}|\mathbf{y}) = \mathcal{N}(\hat{\mathbf{x}}, \Sigma_{\mathbf{x}})$, where $\hat{\mathbf{x}}$ coincides with the optimizer of (1).

In common with the standard approach, the precision matrices $\omega I_p, \Omega$ are assumed to be known. A key advantage of the of Bayesian approach is its ability to infer even these parameters from the data.

3. Prior model for unknown covariance

Consider the following structure of matrix Ω:

$$\Omega = L^\top \Upsilon L, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & 1 & 0 & 0 \\ \mathbf{l}_1 & \vdots & 1 & 0 \\ \vdots & \mathbf{l}_k & \mathbf{l}_{n-1} & 1 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} \upsilon_1 & 0 & 0 & 0 \\ 0 & \upsilon_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \upsilon_n \end{pmatrix}.$$

where the vectors of unknowns are $\mathbf{l}_1, \ldots, \mathbf{l}_{n-1}, \boldsymbol{\upsilon} = [\upsilon_1, \ldots, \upsilon_n]$. The Bayesian formalism requires to define prior distribution on all unknowns. We choose $p(\upsilon_i) = \mathcal{G}(a_0, b_0)$ and

$$p(\mathbf{l}_k|\psi_k) = t\mathcal{N}\left(\mathbf{0}, \psi_k^{-1}, [a_k, b_k]\right), \tag{5}$$

4. Variational LS-APCi algorithm

The formal solution of the estimate is difficult to evaluate:

$$p(\mathbf{x}|\mathbf{y}, M, \upsilon, \mathbf{l}_k) = \int p(\mathbf{y}|\mathbf{x}, M)p(\mathbf{x}|\mathbf{l}_k, \upsilon)p(\mathbf{l}_k, \upsilon)d\mathbf{l}_k d\upsilon. \tag{6}$$

Analytical solution of (6) is not available and a suitable approximation has to be found. Following the Variational Bayes approximation [3], the posterior estimate can be obtained by solving the following equations:

$$\tilde{p}(\mathbf{x}|\mathbf{y}, M) \propto \exp\left(\int \tilde{p}(\upsilon|\mathbf{y})\tilde{p}(\mathbf{l}_k|\mathbf{y}) \log p(\mathbf{x}, \mathbf{y}, \mathbf{l}_k, \upsilon|M) d\upsilon\, d\mathbf{l}_k\right),$$

$$\tilde{p}(\mathbf{l}_k|\mathbf{y}, M) \propto \exp\left(\int \tilde{p}(\mathbf{x}|\mathbf{y})\tilde{p}(\upsilon|\mathbf{y}) \log p(\mathbf{x}, \mathbf{y}, \mathbf{l}_k, \upsilon|M) d\mathbf{x}\, d\upsilon\right),$$

$$\tilde{p}(\upsilon|\mathbf{y}, M) \propto \exp\left(\int \tilde{p}(\mathbf{x}|\mathbf{y})\tilde{p}(\mathbf{l}_k|\mathbf{y}) \log p(\mathbf{x}, \mathbf{y}, \mathbf{l}_k, \upsilon|M) d\mathbf{x}\, d\mathbf{l}_k\right),$$

which can be solved iteratively [3]. The algorithm is called the least square with the prior adaptive covariance with interval ratios restrictions (LS-APCi) algorithm.

4.1 Range enforcement

Since the source term can not be negative, we seek only for positive solutions of the problem. Hence, we may restrict the support of prior $p(x)$ to positive domain only using truncated normal distribution

$$p(x_j) = t\mathcal{N}(0, \sigma_{x_j}^{-1}, \langle 0, \infty \rangle),$$

which moments are non-trivial but available as

$$\widehat{x} = \mu - \sqrt{\sigma}\frac{\sqrt{2}[\exp(-\beta^2) - \exp(-\alpha^2)]}{\sqrt{\pi}(\mathrm{erf}(\beta) - \mathrm{erf}(\alpha))}, \tag{7}$$

where $\alpha = \frac{a-\mu}{\sqrt{2\sigma}}$ and $\beta = \frac{b-\mu}{\sqrt{2\sigma}}$ and $\mathrm{erf}()$ is the error function defined as $\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2} du$ Similarly, the moments for elements of vectors $\mathbf{l}_k$ are computed according to (7) with both-side restrictions.

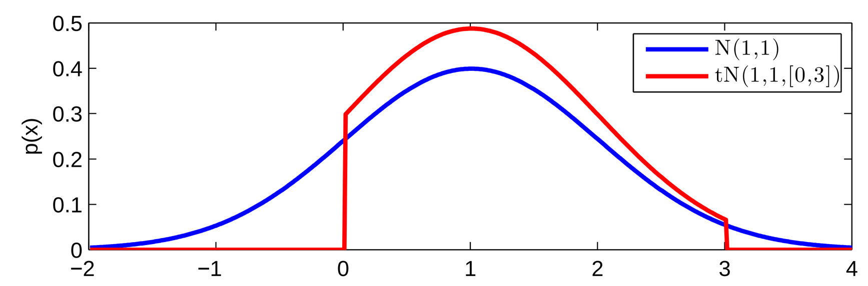

Figure 1: Example of the normal distribution $\mathcal{N}(1, 1)$, blue line, and the truncated normal distribution $t\mathcal{N}(1, 1, < 0, 3 >)$, red line.

5. Optimization approach (CVX toolbox)

For comparison, we solved classical optimization problem formulation (1) using convex optimization toolbox.

The problem (1) together with conditions (2) can be reformulated as

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{X}} \left\{ ||\mathbf{y} - M\mathbf{x}||_2^2 + \alpha g(\mathbf{x}) \right\},$$
$$\text{subject to } \mathbf{x} \geq 0, a_k \leq \frac{\mathbf{x}_1}{\mathbf{x}_k} \leq b_k, \ \forall k, \tag{8}$$

where $\alpha > 0$ is the weight of the regularization term $g(\mathbf{x})$. For comparison, we use Tikhonov regularization and LASSO regularization:

$$g_{\text{Tikhonov}}(\mathbf{x}) = ||\mathbf{x}||_2^2, \tag{9}$$
$$g_{\text{LASSO}}(\mathbf{x}) = ||\mathbf{x}||_1, \tag{10}$$

We use CVX toolbox [1] for convex optimization to obtain solution of formulated optimization problem. Here, limitations of the nuclides ratios (2) can be directly taken into the account using additional condition in the form of "subject to".

The parameter α will be manually selected to achieve the best solution in the forthcoming experiments.

6. Experiment

In this experiment, we consider a simulated release of a mixture of 16 nuclides: Cs-136, Cs-134, Cs-137, I-133, I-131, I-135, I-132, I-134, Kr-85m, Kr-88, Kr-87, Sr-90, Sr-89, Te-132, Xe-135, Xe-133. As a atmospheric transport model we use the Lagrangian particle dispersion model FLEXPART with ECMWF Era-Interim meteorological fields with horizontal resolution 0.5 deg in the case of SRS matrix and 0.25 deg in the case of measurements. The spatial resolution of a model output is approximately 10×10 km. As a release site, the Czech nuclear power plant Temelin is assumed. The Austrian radiation monitoring network comprising of more than 300 receptors is used, see Figure 2, providing GDR measurements with the temporal resolution one hour.

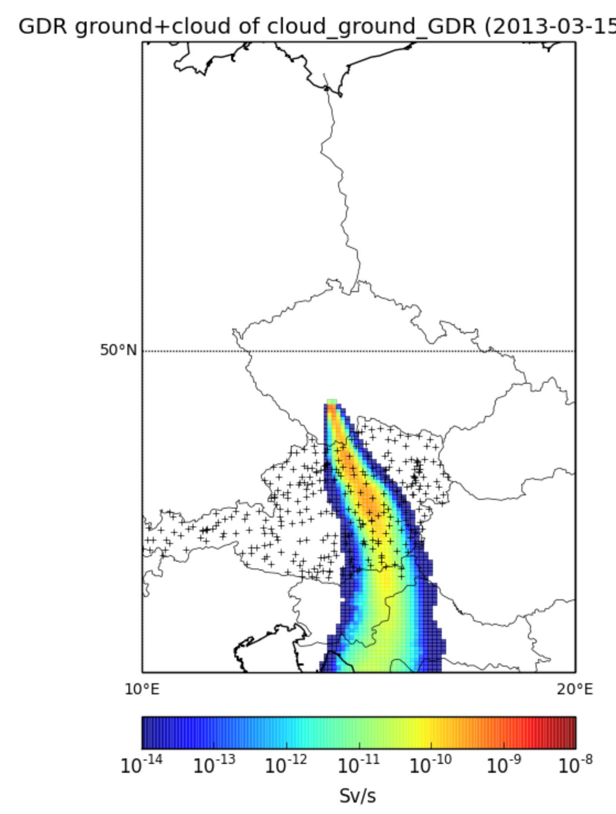

Figure 2: Gamma dose rate from the cloud shine and deposition 12 hours after start of the release.

In this experiment, $\mathbf{y} \in R^{6720\times 1}$ and the total time of measurement is 14 hours, thus, the length of the vector $\mathbf{x}$ is 224 implying $M \in \mathbf{R}^{6720\times 224}$. The simulated release started 4 hours after start of the experiment and lasted for 6 hours followed by another 4 hours with no release. The original releases for each nuclide are displayed using red dashed lines. The ranges of nuclide ratios are selected according to the expert opinion with true ratios laying inside the intervals.

The results are shown in Figure 4 for LA-APCi algorithm and in Figure 5 for CVX constraint optimization with Tikhonov regularization term with manually selected best parameter α. The results from all methods are summarized in Figure 3 using computed normalized mean absolute error (MAE) for each nuclide. We conclude that the results from the LS-APCi algorithm are more robust and reliable then those of the others.

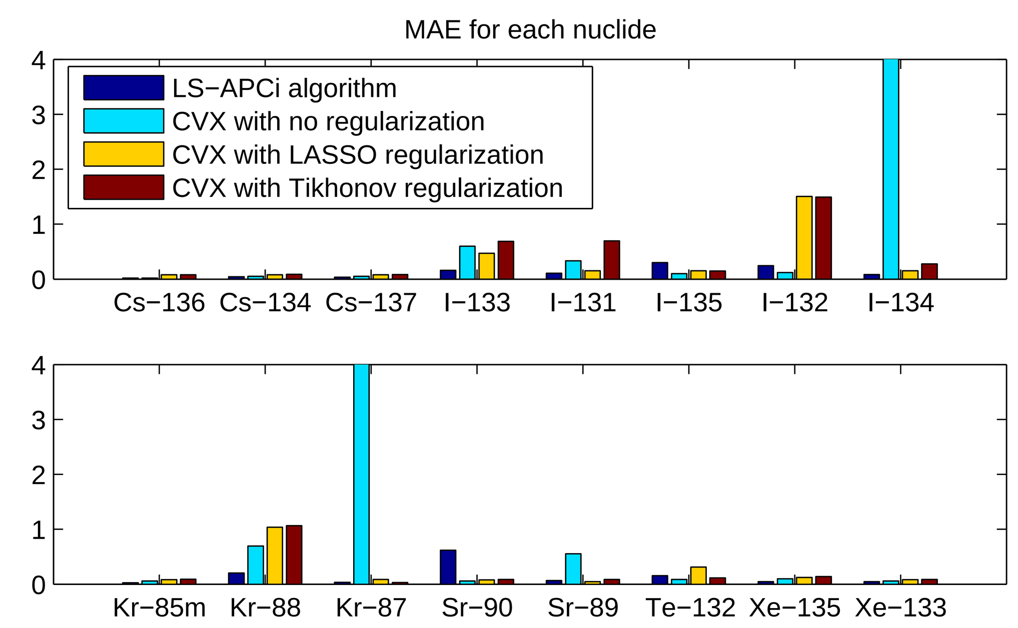

Figure 3: Computed mean absolute error between true source term and the estimated source term for each nuclide for all tested methods.

6.1 Example results

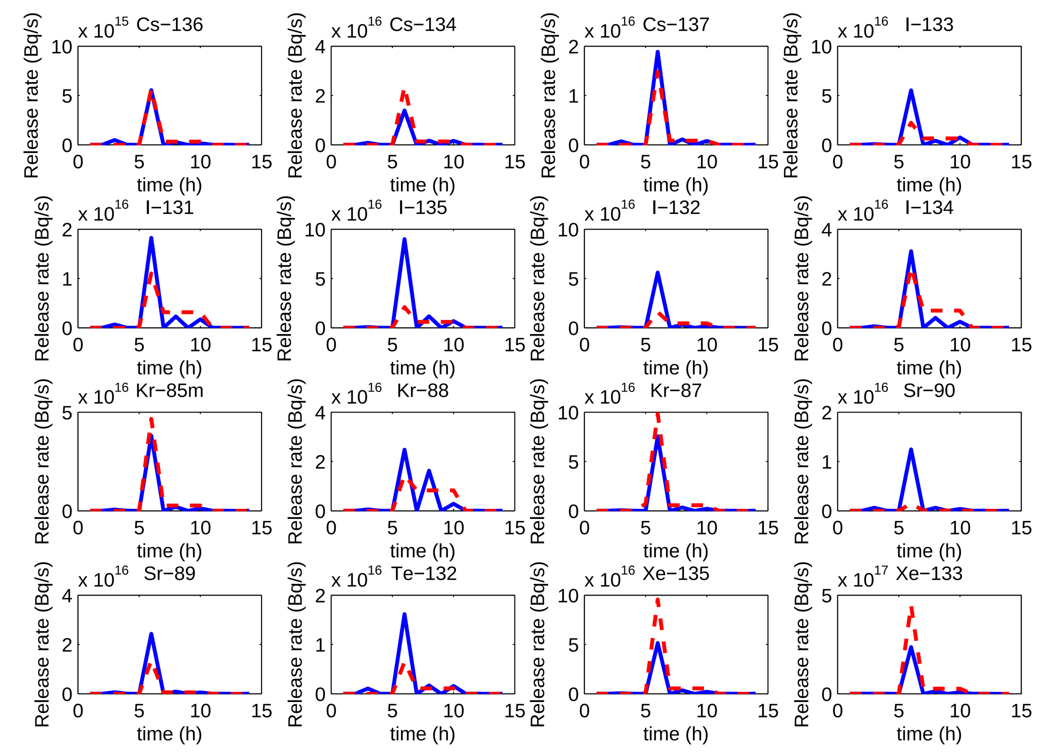

Figure 4: The results of the LS-APCi algorithm for the GDR data containing 16 nuclides. The dashed red lines are true source terms and the solid blue lines are estimated source term by the algorithm.

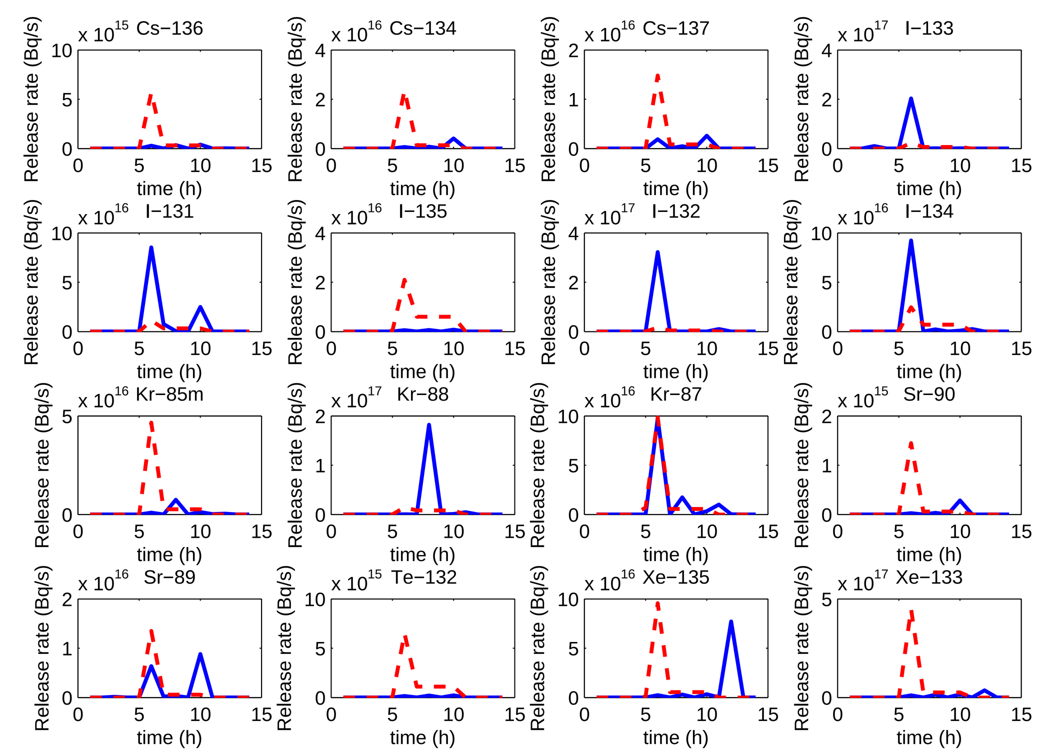

Figure 5: Same as previous but for CVX constraint optimization with Tikhonov regularization with the best manually selected α.

7. Conclusion

- The ill-posed inverse problem with approximately known nuclide ratios is solved.
- This methodology is very flexible and can be also used for many inverse problem formulations.

References

M. Grant and S. Boyd.
CVX: Matlab software for disciplined convex programming, version 2.1, http://cvxr.com/cvx.
2014.

Olivier Saunier, Anne Mathieu, Damien Didier, Marilyne Tombette, Denis Quélo, Victor Winiarek, and Marc Bocquet.
An inverse modeling method to assess the source term of the fukushima nuclear power plant accident using gamma dose rate observations.
Atmospheric Chemistry and Physics, 13(22):11403–11421, 2013.

Václav Šmídl and Anthony Quinn.
The variational Bayes method in signal processing.
Springer Science & Business Media, 2006.

A Stohl, P Seibert, G Wotawa, D Arnold, JF Burkhart, S Eckhardt, C Tapia, A Vargas, and TJ Yasunari.
Xenon-133 and caesium-137 releases into the atmosphere from the fukushima dai-ichi nuclear power plant: determination of the source term, atmospheric dispersion, and deposition.
Atmospheric Chemistry and Physics, 12(5):2313–2343, 2012.

This research is supported by EEA/Norwegian Financial Mechanism under project 7F14287 STRADI: Source-Term Determination of Radionuclide Releases by Inverse Atmospheric Dispersion Modelling. Poster created with `beamerposter`.

WWW: *http://stradi.utia.cas.cz/*